# Corrected Mathematical Models for Motions of the Gyroscope with one Side Free Support

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**Abstract:** The recent publications about gyroscope effects explained their physics and described them by mathematical models based on the action of forces and inertial torques of classical mechanics. This new analytical approach finally solved the old problem of the dynamic of rotating objects and showed their kinetic energy is the base of gyroscopic effects. Gyroscopic effects result from the action of the two sets of interrelated inertial torques acting about two axes. Each set contains torques generated by the centrifugal, Coriolis forces, and the change in the angular momentum. Detailed study of the inertial torque of the centrifugal force showed its expression derived with an error of mathematical processing. This error gives a less value for the angular velocity of the slow rotation of the gyroscope about one axis that cannot be measured. The angular velocity of the fast rotation of the gyroscope about the other axis is measured but remains of the same value that gives the expression of the torque with error. This manuscript presents the corrected mathematical model for the motion of the gyroscope suspended from the flexible cord.

**Keywords:** Gyroscope theory, Inertial torque, Angular velocities, Mathematical model.

## INTRODUCTION

The solutions to gyroscope problems were problematic for a long time and known publications do not a clear picture of the physics gyroscopic effects [1-4]. The partial solution of the gyroscope properties does not describe completely entire their physics [5-8]. Stubborn researchers are continuing to find answers to gyroscope problems [9-11]. The new study of gyroscopic effects yields the expressions for the inertial torques generated by the centrifugal, Coriolis forces, and the change in the angular momentum of the spinning objects [12]. These inertial torques are interrelated and constitute one system with two sets acting about two axes of gyroscope motions. Each set contains four torques mentioned above where the torque of the centrifugal force acts about two axes. The motions of the gyroscope are synchronized and expressed by the defined dependency of the angular velocities. The precessed velocity about one axis is many times bigger than the other one and can be measured by laboratory devices [13-16]. Practical tests of the gyroscope motions suspended from the flexible cord validated the angular velocity of the precession rotation [17]. The measure of the slow rotation of the gyroscope about the other axis was problematic. The dynamical process, very small angle of rotation, and drop of the revolutions of the gyroscope rotor did not give the ability to measure the angular velocity about another axis.

Detailed study of the inertial torque generated by the centrifugal force of rotating mass showed its expression derived with an error of mathematical processing. This error is related to the incorrect limit of the integral for the resistance inertial torque of the centrifugal forces, which value increased twice and reflected on the double decrease of the angular velocity around one axis. The angular velocity around the other axis remains the same because the coefficient of the dependency of the angular velocities is increased twice and compensates for the double increase of the angular velocity of rotation about the first axis. The equality of the kinetic energies about two axes is the physical principle of the dependency of the angular velocities of the gyroscope rotation about two axes that described several publications [13, 16]. The renovated system of the inertial torques and the dependency of the angular velocities of the gyroscope is presented in Table **1**.

**Table 1: Inertial Torques of the Gyroscope of Horizontal Disposition and the Dependency of the Angular Velocities**

| Inertial Torques Generated by | Action | Equation |
|---|---|---|
| Centrifugal forces | Resistance | $T_{ct.i} = (4\pi^2/9)J\omega\omega_i$ |
| | Precession | |
| Coriolis forces | Resistance | $T_{cr.i} = (8/9)J\omega\omega_i$ |
| Change in an angular momentum | Precession | $T_{am.i} = J\omega\omega_i$ |
| Dependency of the angular velocities: $\omega_\gamma = (8\pi^2 + 17)\omega_x$ | | |

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where $\omega_i$ is the angular velocity about axis *i;* $\omega$ is the angular velocity about axis *oz*; *J* is the moment of inertia of the gyroscope spinning rotor.

The renovated expressions for the inertial torques of the gyroscope and the dependency of the angular velocities are used for mathematical modeling of motions of the gyroscope suspended from the flexible cord.

## METHODOLOGY

The study of the action of the forces and motions of the gyroscope suspended from the flexible cord is typical laboratory work by the course of engineering mechanics at the universities. The theoretical part of the laboratory work contains the mathematical model of motions for the gyroscope that was considered in detail in the publication with an error of expression of the centrifugal inertial torque [17]. The renovated mathematical model for motions of the gyroscope of horizontal disposition was modified and presented by Eqs. (1) – (3). The components of the inertial forces generated by the rotating center mass of the gyroscope were removed because they have small values of a high order. The force and torques acting on the gyroscope suspended from the flexible cord are demonstrated in Cartesian 3D coordinate system Σ*oxyz* (Figure **1**). The symbolic components of Figure **1** are presented in Euler's differential equations for gyroscope motions based on all acting torques about axes ox and oy. The secondary inertial torques of centrifugal forces of precessed motions about axes *oy* and *ox* are subtracted by the rules of mathematics because they have the same expressions in Eqs. (1) and (2) [13].

$$J_x \frac{d\omega_x}{dt} = T - T_{ct.x} - T_{cr.x} - T_{am.y} \tag{1}$$

$$J_y \frac{d\omega_y}{dt} = T_{ct.x} + T_{am.x} - T_{cr.y} \tag{2}$$

$$\omega_\gamma = (8\pi^2 + 17)\omega_x \tag{3}$$

where $J_x = J_y$ is the mass moment of gyroscope inertia about axis *ox* and *oy* respectively that calculated by the parallel axis theorem; $\omega_x$ and $\omega_y$ is the angular velocity of precession about axis *ox* and *oy* respectively; $T = Wl = Mgl$ is the torque generated by the gyroscope weight *W* about axis *ox; M* is the mass; *g* is the gravity acceleration; *t* is the time; $T_{ct.x}$ is the resistance and precession torque generated by centrifugal forces about axis *ox* and *oy* respectively (Table **1**); $T_{cr.x}$ and $T_{cr.y}$ is the resistance torque generated by Coriolis forces about axis *ox* and *oy* respectively (Table **1**); $T_{am.x}$ and $T_{am.y}$ is the precession and resistance torque that generated by the change in the angular momentum of the spinning rotor about axis *ox* and *oy,* respectively (Table **1**).

Substituting defined parameters and equations of the inertial torques (Table **1**) into Eqs. (1) and (2) and

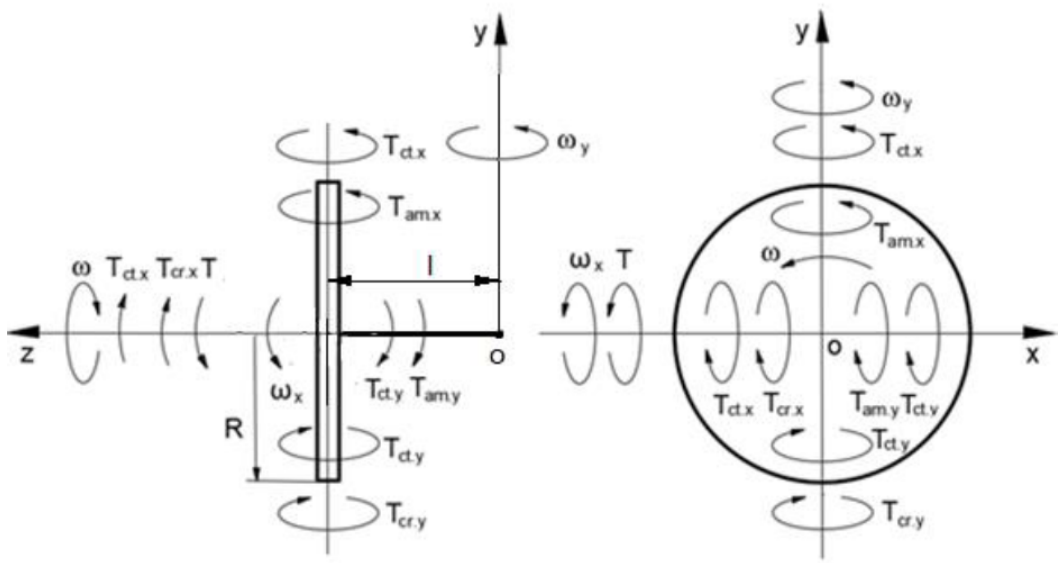


**Figure 1:** Schematic of acting torques on the gyroscope suspended from the flexible cord.

transformation yield the following system of differential equations:

$$J_x \frac{d\omega_x}{dt} = Mgl - \frac{4}{9}\pi^2 J\omega\omega_x - \frac{8}{9}J\omega\omega_x - J\omega\omega_y \quad (4)$$

$$J_y \frac{d\omega_y}{dt} = \frac{4}{9}\pi^2 J\omega\omega_x + J\omega\omega_x - \frac{8}{9}J\omega\omega_y \quad (5)$$

where *l* is the overhang distance of the gyroscope center-mass to the free support; parameters of the mass moment of inertia of the gyroscope are represented in Table **2**, and other components are as specified above.

**Table 2: Mass Moments of Inertia for the Gyroscope Components**

| Title | Equation |
|---|---|
| The gyroscope mass moment of inertia of the gyroscope about the axis *ox* and *oy* | $J_x = J_y = (MR^2/4) + Ml^2$ |
| The rotor's mass moment of inertia about axis *ox* and *oy* | $J_x = J_y = (m_r R^2/4)$ |
| The rotor's mass moment of inertia about axis *oz* | $J = (m_r R^2/2)$ |

where $m_r$ is the mass of rotating parts, *R* is the conventional radius of the disc-type rotor.

Expression of $\omega_y$ of Eq. (3) is substituted into the last component of Eq. (4) with $\omega_y$ and transformation yield the equation of the gyroscope motion about axis *ox*:

$$J_x \frac{d\omega_x}{dt} = Mgl - \left(\frac{76\pi^2 + 161}{9}\right) J\omega\omega_x \quad (6)$$

where all parameters are as specified above.

Solution of Eq. (6) and Eq. (3) gives the angular velocities of rotation $\omega_x$ and $\omega_y$, respectively.

## WORKING EXAMPLE

The mathematical model for motions of the gyroscope suspended from the flexible cord is considered for the Super Precision Gyroscope, "Brightfusion LTD", which technical data is presented in Table (**3**). Substituting parameters from Table **3** into the expression of the precession torque of the change in the angular momentum and computing yields:

$$T_{am.x} = J\omega\omega_{p.i} = 0{,}5726674 \times 10^{-4}\omega\omega_{p.i} = \\ 0{,}5726674 \times 10^{-4} \times 10^4 \times (2\pi / 60)\omega_x = 0{,}059969\omega_x \ \text{Nm} \quad (7)$$

where all parameters are as specified above.

Substituting defined gyroscope parameters represented in Table **3** into Eq. (6) yields the following differential equation:

$$1{,}9974649 \times 10^{-4} \frac{d\omega_x}{dt} = 0{,}146 \times 9{,}81 \times 0{,}0325 - 101{,}232214 \times 0{,}059969\omega_x \quad (8)$$

Simplifications of Eq. (8) and transformation brings the following equation:

$$3{,}290253 \times 10^{-5} \frac{d\omega_x}{dt} = 0{,}007667 - \omega_x \quad (9)$$

Separating variables and transformation for the differential Eq. (9) gives the following equation:

$$\frac{d\omega_x}{0{,}007667 - \omega_x} = 30392{,}796dt \quad (10)$$

Presenting Eq. (10) into integral forms at definite limits yields the following integral equations:

**Table 3: Technical Data of Super Precision Gyroscope, "Brightfusion LTD"**

| | | | |
|---|---|---|---|
| Rotating components | | Mass, kg | 0,1159 |
| | | Angular velocity, $\omega$ | 10000,0 rpm |
| Gyroscope | | Mass, *M*, kg | 0,146 |
| | | Overhang, *l* m | 0,0325 |
| Mass moment of inertia kgm$^2$ | About axis *oz, J* | Rotating components | $0{,}5726674 \times 10^{-4}$ |
| | About axis *ox* and *oy*, $J_x = J_y$ | Gyroscope | $1{,}9974649 \times 10^{-4}$ |

$$\int_0^{\omega_x} \frac{d\omega_x}{0{,}007667 - \omega_x} = 30392{,}796 \int_0^t dt \tag{11}$$

The left integral of Eq. (11) is tabulated and represented the integral $\int \frac{dx}{a-x} = -\ln|a-x| + C$. The right integral is simple and integrals have the following solution:

$$-\ln\left(0{,}007667 - \omega_x\right)\Big|_0^{\omega_x} = 30392{,}796t\Big|_0^t$$

that gave rise to the following

$$1 - \frac{\omega_x}{0{,}007667} = e^{-30392{,}796t} \tag{12}$$

Solving of Eq. (12) gives the equation for the precession angular velocity of the gyroscope around axis *ox:*

$$\omega_x = 0{,}007667 \times (1 - e^{-30392{,}796t}) \tag{13}$$

The expression $e^{-30392{,}796t}$ of Eq. (13) has a small value of high order and can be neglected. The computed angular velocity of the gyroscope motion about axis *ox* gives the following result:

$$\omega_x = 0{,}007667 \quad rad/s = 0{,}439^o/s \tag{14}$$

Substituting Eq. (14) into Eq. (3) and transformation yields the following result of the gyroscope precession angular velocity around axis *oy*:

$$\omega_y = (8\pi^2 + 17)\omega_x = (8\pi^2 + 17) \times 0{,}007667 = 0{,}735751 \quad rad/s = 42.155^o/s \tag{15}$$

The time spent on one revolution about axis *oy* takes $t = 2\pi/\omega_y = 360^0/42{,}155^o/s = 8{,}539$ s. The precession angular velocity about the axis *ox* is $0.746^o$ /s. The time spent on the turn of $20^o$ axis *ox* ($\pm 10^0$ about the horizontal location) takes $t = 2\pi/\omega_x = 20^o/0{,}439^o$ /s = 45,558 s. The measurement of the time of the gyroscope motion about axis *ox* is additional and not main due to the drop of the velocity of the spinning disc.

## RESULT AND DISCUSSIONS

Mathematical models for the motions of the gyroscope suspended from the flexible cord are presented in the Cartesian 3D coordinate system Σ*oxyz* (Figure **1**). The axel of the rotor spinning with the angular velocity *ω* coincides with axis *oz*. The torque of the gyroscope weight acts around axis *ox* with its slow rotation. The precessed motion of the gyroscope is around axis *oy*.

The corrected mathematical models for the gyroscope motions around axes *ox* and *oy* were tested and the results are represented in Table **4**.

**Table 4: Experimental and Theoretical Results of the Gyroscope Motions**

| Gyroscope Average Parameters | Test | Theoretical | Difference |
|---|---|---|---|
| Time of precession (one revolution) about axis *oy* | 8,1 s | 8,539 s | 5.141% |
| Time of rotation about axis *ox* | | $0{,}439^o$/s | |

The result of the theoretical calculation and practical test of the gyroscope (Table **4**) are the same for the precessed motion about axis *oy* in publications [17]. The time of the gyroscope rotation on $20^o$ about axis ox is almost twice less than in the first publications [13, 17]. The measurement of the time of the rotation about axis *ox* was problematic practically and not implemented due to the very small value of the angular velocity on the movable gyroscope. The result of the precessed motion about axis *oy* has served as the basis for the first publications. Expanded tests of the renovated mathematical models for gyroscope motions can additionally validate their correctness. The study of the acceptable differences of theoretical and practical results represented in publications and recommendations of the experts demonstrate the permissible discrepancy should not exceed 10% [18].

## CONCLUSION

Analytical solution for gyroscopic effects is a sophisticated process that is linked to complex mathematical models and their processing. In such cases, omissions in solutions and following corrections are inevitable. This statement is confirmed by the new solutions to gyroscopic effects, publications, and criticism of mistakes. The error in the expression of the one inertial torque generated by the gyroscope is not fundamental, but can yield distorted results. The corrected mathematical model for the inertial torque in the aggregate with others was validated by practical tests of the mathematical models for gyroscope motions and can be used for solutions to gyroscope problems in engineering.

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