Robust Global Sensitivity Analysis for Robust Design under Parameter Uncertainty

Zheng Mingliang*

Mechanical and Electrical Engineering of Wuxi Taihu University, 214064, Wuxi, China

Abstract: Based on the theory and method of robust design, the robust global sensitivity analysis of products or systems under parameter uncertainty is discussed. A basic idea of the author is to define the robust sensitivity that is the importance measure of the design variables for product functional response function distribution. The Taylor series of moments of the functional response function is carried out, and the approximate analytical formulas of robust global sensitivity are obtained by using the importance measure model based on variance. Finally, a numerical example is given to illustrate the operation principle of this method, and an engineering example is given to verify the correctness of this method.

Keywords: Robust design, Robust sensitivity, Importance measure, Variance model.

1. INTRODUCTION

The sensitivity of functional response function of product or system to input design parameters and uncertaintyies is called sensitivity analysis, which is widely used in engineering design. There are two main types: local sensitivity analysis and global sensitivity analysis [1]. In robust design, robust sensitivity analysis can help to understand the relative importance of design parameters affecting product functional robustness, so that the design parameters with high importance can be prioritized and with low importance can be neglected in design and optimization. At present, there are many models to describe uncertainty: stochastic model, fuzzy model, non-probabilistic model and cross-mixing model.

At present, the research on robustness sensitivity of parameters at home and abroad usually defines the partial derivative of robustness index to subjective design parameters. In this definition, the derivative value is the nominal value of subjective design parameters and when considering the sensitivity of one parameter, the influence of other parameters can not be considered. Its essence is local sensitivity. For global sensitivity analysis method, there are mainly non-parametric method proposed by Saltellid and Helton [2], variance analysis method proposed by Sobol [3], and moment-independent analysis method proposed by Borgonovo [4]. Taguchi robust design is based on factor level table of experimental scheme, and the importance of design parameters to robustness index is ranked. Generally, the computation is larger.

In this paper, the robust sensitivity of robust design is defined as the importance measure of the design variables for product functional response function distribution. By second-order Taylor expansion, the unconditional variance and conditional expected variance of functional response function are obtained. According to the importance measure model based on variance, the formula for calculating global robust sensitivity is given, and an effective robust design way is pointed out. Finally, an example is given to illustrate the correctness and simplicity of the proposed method.

2. THE RELATION BETWEEN ROBUST SENSITIVITY AND IMPORTANCE MEASURE OF PARAMETERS IN ROBUST DESIGN

Robust design was first proposed by Dr. Taguchi in the early 1970s [5]. According to axiomatic design theory, the process of design can be regarded as the process of mapping from functional domain to physical domain. For products and systems, a functional response function is

$$Y = g(X_1, X_2 ... X_n) = g(X_1^* + \Delta X_1, X_2^* + \Delta X_2 ... X_n^* + \Delta X_n) \tag{1}$$

Here Y is functional response function, $X_1, X_2 ... X_n$ are uncertain parameters, $X_1^*, X_2^* ... X_n^*$ are nominal values, $\Delta X_1, \Delta X_2 ... \Delta X_n$ are uncontrollable factores.

At present, there are three methods to measure the robustness of the functional response function in robust optimization design: (1) the variance; (2) quantile difference proposed by Du [6]; (3) the concept of Shannon entropy proposed by Beer [7]. We adopt (1) as the robustness.

*Address correspondence to this author at the Mechanical and Electrical Engineering of Wuxi Taihu University, 214064, Wuxi, China;
E-mail: zhmlwxcstu@163.com

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The importance measure of basic variables can be defined as the contribution of the uncertainty of input variables to the uncertainty of output response [8-9]. Which is also called global sensitivity. The importance measure index can provide the ranking of the influence of random variables on response, and can be used for design or optimization. The importance measure is applied to the robust design, which the key is to use robustness index (1), then the robust global sensitivity of parameter can be regarded as an important measure, so the importance measure index S_l based on variance is:

$$S_l = \frac{Var(E(Y|\boldsymbol{X}_l)}{Var(Y)} \tag{2}$$

Var($E(Y|\boldsymbol{X}_l)$) is conditional expectation variance only when $\boldsymbol{X}_l$ is nominal value. 0≤ S_i ≤1, S_i is bigger show that the influence of variables $\boldsymbol{X}_i$ on variance of Y is greater.

3. ROBUST GLOBAL SENSITIVITY COMPUTATION BASED ON IMPORTANCE MEASURE MODEL OF VARIANCE

Formula (1) is expanded at the nominal value $\boldsymbol{X}^*$ according to the second-order taylor series:

$$Y = g(X_1, X_2 ... X_n) \approx g(X_1^*, X_2^* ... X_n^*) + \sum_{i=1}^{n} \frac{\partial g}{\partial X_i}\Big|_{X^*}(\Delta X_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial g}{\partial X_i \partial X_j}\Big|_{X^*}(\Delta X_i \cdot \Delta X_j) \tag{3}$$

From the knowledge of probability theory, if $\Delta X_i, \Delta X_j$ are independent, $f(\Delta X_i), g(\Delta X_j)$ are also independent, but $\Delta X_{i,} \Delta^2 X_i, \Delta X_i \Delta X_j$ are not independent. The unconditional variance of Y is:

$$Var(Y) \approx \sum_{i=1}^{n} (\frac{\partial g}{\partial X_i}\Big|_{X^*})^2 var(\Delta X_i) + \frac{1}{4}\sum_{i=1}^{n} (\frac{\partial^2 g}{\partial^2 X_i}\Big|_{X^*})^2 var(\Delta^2 X_i) + \sum_{1 \le i < j \le n} [(\frac{\partial^2 g}{\partial X_i \partial X_j})\Big|_{X^*}]^2 var(\Delta X_i \Delta X_j) \tag{4}$$

The expectation of Y is:

$$E(Y) \approx g(\boldsymbol{X}^*) + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 g}{\partial^2 X_i}\Big|_{X^*} \sigma_i^2 \tag{5}$$

The conditional expectation variance of Y when X_i is X_i' :

$$Var(E(Y|X_i)) \approx [\frac{\partial g}{\partial X_i}\Big|_{X^*}]^2 \sigma_i^2 + \frac{1}{2}[\frac{\partial^2 g}{\partial^2 X_i}\Big|_{X^*}]^2 \sigma_i^4 \tag{6}$$

The (4) include local sensitivity $\frac{\partial g}{\partial X_i}\Big|_{X^*}$ and $\frac{\partial^2 g}{\partial X_i \partial X_j}\Big|_{X^*}$, which depend on the nominal value of design parameters, it belongs to a relatively static sensitivity.

$(\boldsymbol{X}_i - \boldsymbol{X}_i^*)^T$ and $(\boldsymbol{X}_i - \boldsymbol{X}_i^*)^T(\boldsymbol{X}_i - \boldsymbol{X}_i^*)$ reflect the deviation from nominal value.

$\sum_{i=1}^{n} \frac{\partial g}{\partial X_i}\Big|_{X^*}(\Delta X_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial g}{\partial X_i \partial X_j}\Big|_{X^*}(\Delta X_i \cdot \Delta X_j)$ is sensitivity, it belongs to a global sensitivity. The robust global sensitivity of parameter X_i in robust design is:

$$S_i = \frac{Var(E(Y|X_i)}{Var(Y)} = \frac{[\frac{\partial g}{\partial X_i}\Big|_{X^*}]^2 \sigma_i^2 + \frac{1}{2}[\frac{\partial^2 g}{\partial^2 X_i}\Big|_{X^*}]^2 \sigma_i^4}{\sum_{i=1}^{n}(\frac{\partial g}{\partial X_i}\Big|_{X^*})^2 \sigma_i^2 + \frac{1}{2}\sum_{i=1}^{n}(\frac{\partial g}{\partial X_i}\Big|_{X^*})^2 \sigma_i^4 + \sum_{1 \le i < j \le n}(\frac{\partial^2 g}{\partial X_i \partial X_j}\Big|_{X^*})^2 \sigma_i^2 \sigma_j^2} \tag{7}$$

So the robust global sensitivity vector of design parameters is $\boldsymbol{R}^s = (S_1, S_2, ... S_n)^T$, which is a function of $\boldsymbol{X}^*$. Compared with the traditional importance measure, the robust global sensitivity vector proposed in (7) may not be satisfied $\|\boldsymbol{R}^s(\boldsymbol{X}^*)\|_1 = 1$.

The content of robust design includes: on the one hand, to make the mean value of functional response function as close as possible to the expected target, on the other hand, to minimize the variance of functional response function, which is regarded as a process of optimizing the nominal value of product design parameters. If we minimize the robust global sensitivity of parameters with the greatest importance measure, the fluctuation of functional response function is actually minimized, so the robust design can also be expressed as the following optimal design model:

$$\begin{aligned} & min \quad \|\boldsymbol{R}^s(\boldsymbol{X}^*)\|_\infty \\ & subject\ \ to \quad \begin{cases} g(\boldsymbol{X}^*) + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 g}{\partial^2 X_i}\Big|_{X^*} \sigma_i^2 = T \\ h_i(\boldsymbol{X}^*) = 0 (i = 1, 2 ... m) \\ w_j(\boldsymbol{X}^*) \le 0 (j = 1, 2 ... l) \\ \boldsymbol{X}^L \le \boldsymbol{X}^* \le \boldsymbol{X}^U \end{cases} \end{aligned} \tag{8}$$

We can also make the robust feasibility analysis of constraint functions h_i, w_j to modify the optimization model. Different from the general robust optimization design, the robust optimization process based on the importance measure analysis can actually simultaneously complete the nominal value of parameters and robust global sensitivity of parameters, which is parallelized and efficient.

4. EXAMPLE ANALYSIS

4.1. Numerical Example

Nonlinear functional response function: $g(x)=x_2^2-3x_1+x_2$, x_1,x_2 are independent and obey the normal distribution (μ= 4, σ =1). The unconditional variance and the conditional expected variance under the action of a single parameter are as follows:

$$var(Y)\approx 81+9+2=92\ ,\quad var(E(Y|x_1)\approx 9\ ,$$

$$var(E(Y|x_2)\approx 82 \tag{9}$$

The rank of robust global sensitivity is $S_2^M > S_1^M$. By changing the nominal value μ of maximum sensitivity parameters, the variability of the functional response function can be effectively reduced and the robustness of product design can be improved.

4.2. Engineering Example

The schematic diagram of the pneumatic commutator in reference [10] is shown in Figure **1**. The system will complete the reversing action in t = 1s with load and resistance F. According to the analysis, the final speed of piston commutation:

$$v=\sqrt{\frac{(\frac{\pi}{2}D^2p_w-2F)Lg}{F_G}} \tag{10}$$

Here D is diameter of piston, L is stroke of piston, F_G is gravity load, p_w is cylinder pressure, in this case, the functional response function is v, F=250, g=10, F_G,D,L,p_w are uncertainties, the statistical characteristics are:

$$\sigma^2(D)=\sigma^2(L)=\sigma^2(p_w)=180\times10^{-6}\ ,$$
$$F_G^*=900,\sigma^2(F_G^*)=20 \tag{11}$$

The results of robust optimal design analysis in reference [10] are:

The global robustness sensitivity of design parameters is solved based on importance measure:

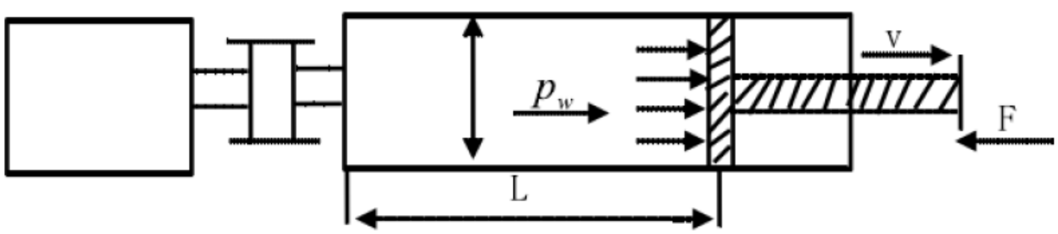

Figure 1: Schematic diagram of pneumatic commutator.

Table 1: **The Result of Robust Design Optimization for Pneumatic Commutator**

Name		Optimum result
Function	v (mm*s^{-1})	Expectation 960、variance 4.54
Design parameters	D/mm	25
	L/mm	51.63
	p_w/Mpa	3.20
Sensitivity	D is 0.68、L is 0.28、 p_w is (-3.88)	

$$\begin{aligned}\Delta v \approx & \frac{Ax_1x_2x_3}{x_4v(\boldsymbol{X})}\Big|_{X^*}\Delta x_1+\frac{Ax_1^2x_3-B}{2x_4v(\boldsymbol{X})}\Big|_{X^*}\Delta x_2+\frac{Ax_1^2x_2}{2x_4v(\boldsymbol{X})}\Big|_{X^*}\Delta x_3 \\ & -\frac{Ax_1^2x_2x_3-Bx_2}{2x_4^2v(\boldsymbol{X})}\Big|_{X^*}\Delta x_4+\frac{1}{2}\{\frac{Ax_2x_3[v(\boldsymbol{X})-x_1v_1(\boldsymbol{X})]}{x_4v^2(\boldsymbol{X})}\Big|_{X^*}\Delta^2x_1 \\ & -\frac{(Ax_1^2x_3-B)v_2(\boldsymbol{X})}{2x_4v^2(\boldsymbol{X})}\Big|_{X^*}\Delta^2x_2--\frac{Ax_1^2x_2v_3(\boldsymbol{X})}{2x_4v^2(\boldsymbol{X})}\Big|_{X^*}\Delta^2x_3+ \\ & \frac{(Ax_1^2x_2x_3-Bx_2)(2x_4v(\boldsymbol{X})+x_4^2v_4(\boldsymbol{X}))}{2x_4^4v^2(\boldsymbol{X})}\Big|_{X^*}\Delta^2x_4\}+ \\ & \frac{Ax_1x_3[v(\boldsymbol{X})-x_2v_2(\boldsymbol{X})]}{x_4v^2(\boldsymbol{X})}\Big|_{X^*}\Delta x_1\Delta x_2+\frac{Ax_1x_2[v(\boldsymbol{X})-x_3v_3(\boldsymbol{X})]}{x_4v^2(\boldsymbol{X})}\Big|_{X^*}\Delta x_1\Delta x_3- \\ & \frac{Ax_1x_2x_3[v(\boldsymbol{X})+x_4v_4(\boldsymbol{X})]}{x_4^2v^2(\boldsymbol{X})}\Big|_{X^*}\Delta x_1\Delta x_4+\frac{Ax_1^2[v(\boldsymbol{X})-x_2v_2(\boldsymbol{X})]}{2x_4v^2(\boldsymbol{X})}\Big|_{X^*}\Delta x_2\Delta x_3- \\ & \frac{(Ax_1^2x_3-B)[v(\boldsymbol{X})-x_2v_2(\boldsymbol{X})]}{2x_4^2v^2(\boldsymbol{X})}\Big|_{X^*}\Delta x_2\Delta x_4-\frac{Ax_1^2x_2[v(\boldsymbol{X})+x_4v_4(\boldsymbol{X})]}{2x_4^2v^2(\boldsymbol{X})}\Big|_{X^*}\Delta x_3\Delta x_4\end{aligned} \tag{12}$$

$$Here: A=\frac{\pi}{2}g, B=2Fg, D, L, p_w, F_G = x_1, x_2, x_3, x_4.$$

$v_i(\boldsymbol{X})$ Represents partial derivative of x_i. The second-order and cross-higher-order terms are omitted and the central expectation value of each parameter is 0. The result is Table **2**:

It can be seen that the ranking result of robust global sensitivity vector $R(\boldsymbol{X}^*)$ is consistent with the result of robust sensitivity matrix in reference [10]. Under the constraints of the designed system, the robustness of the pneumatic commutator is mainly affected by p_w, the control of variation of cylinder pressure is the main factor to realize the robustness.

5. CONCLUSION

In this paper, the importance measure theory is introduced into robust design for global robust sensitivity analysis under parameter uncertainties. A formula for calculating robust sensitivity is given in (2), it takes advantage of variance ; Robustness Index of nonlinear functional response function is given in (4) by Stochastic theory ; Robust optimization design model with robust sensitivity as objective function is given in (8) by norm criterion. At the same time, it is noted that this paper only considers robust design of single-objective functional and assumes the design parameters are independent. The importance measure model based on variance sometimes causes the positive and negative conditional variances of functional response function to cancel each other.

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Table 2: Robust Global Sensitivity Analysis Results of Pneumatic Commutator

Name	D	L	p_w	F_G	Sum	Error
S_i^M	0.075	0.026	0.8736	0.008	98.16%	1.84%

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