Remarks on Solving Methods of Nonlinear Equations

Sha Huang[1], and Yaping Zhao[1,*]

[1]*School of Mechanical Engineering and Automation, Northeastern University, Shenyang 110819, China*

Abstract: In the field of mechanical engineering, many practical problems can be converted into nonlinear problems, such as the meshing problem of mechanical transmission. So the solution of nonlinear equations has important theoretical research and practical application significance. Whether the traditional Newton iteration method or the intelligent optimization algorithm after the popularization of computers, both them have been greatly enriched and developed through the continuous in-depth research of scholars at home and abroad, and a series of improved algorithms have emerged. This paper mainly reviews the research status of solving nonlinear equations from two aspects of traditional iterative method and intelligent optimization algorithm, systematically reviews the research achievements of domestic and foreign scholars, and puts forward prospects for future research directions.

Keywords: Nonlinear equations, Iterative method, Optimization method, Intelligent algorithm, The optimal solution, Gear meshing.

1. INTRODUCTION

In the field of mechanical engineering, the problem of solving nonlinear equations has become one of the most important research topics. Many practical problems can be converted to solving nonlinear equations, so its research has important theoretical significance and application value.

For example, in the research of meshing theory of mechanical transmission, it is often necessary to determine some characteristic points on the tooth surface to determine the processing and basic technical parameters of the transmission pair. Due to the nonlinear nature of the tooth surface, it is inevitable to encounter the problem of solving nonlinear equations to determine the parameters contained in the problem.

The nonlinear equations generally refer to a nonlinear equation system composed of several nonlinear equations [1], which can be expressed in the following form:

$$\begin{cases} f_1(x_1, x_2, \cdots x_n) = 0 \\ f_2(x_1, x_2, \cdots x_n) = 0 \\ \qquad \vdots \\ f_m(x_1, x_2, \cdots x_n) = 0 \end{cases} \tag{1.1}$$

where m represents the number of equations, n represents the number of unknowns. $f_i(i = 1, 2, \cdots m)$ is a real-valued function defined on a region D which is in a m dimensional Euclidean Space R^m.

when $m > n$, the equations only be solved in the sense of optimization, can't get a exact solution. When $m = n$, the exact solution can be get if it exist. When $m < n$, the solutions of the equations are generally hypersurfaces or hypercurves in a high dimensional space.

By introducing vector notation, the above equation can also be expressed as:

$$F(x) = 0 \tag{1.2}$$

Here, $F(x)$ represents a nonlinear mapping on the region $D \subset R^{\mathrm{n}} \to R^{\mathrm{n}}$.

Many practical problems can be transformed into nonlinear equations, and analytic solutions can be obtained for some special equations. However, due to the nonlinearity of the equations, it is only impossible to judge if the solution exist, and it is difficult or impossible to obtain the numerical solution. In view of this, domestic and foreign scholars have done a lot of research, and put forward plenty of methods to solve it.

The current solution methods can be divided into two categories: direct method and optimization method [2, 3].

The so-called direct method is to use the idea of iteration to construct iterative functions, and then solve the equations, the most representative is Newton method [4, 5], its most prominent advantage is that when the initial point is close enough to the solution of the problem, it can get a good convergence effect. However, Newton's method also has obvious shortcomings:

*Address correspondence to this author at the School of Mechanical Engineering and Automation, Northeastern University, Shenyang 110819, China; E-mail: zhyp_neu@163.com

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(1) When the initial point is not close enough to the solution, the convergence rate of Newton's method is slow or even not converge;

(2) It is necessary to calculate the Jacobian matrix and solve a linear system of equations in each iteration. When the scale of the problem is large, it is difficult to solve Newton's equation accurately;

(3) When the Jacobian matrix is singular or close to singular, the Newtonian direction is not defined and the iterative method cannot continue.

In view of the shortcomings of Newton's method, many improved algorithms have been proposed on the basis of Newton's method [6-8]. Typical methods include pseudo-Newton method, secant method (discrete Newton method), finite difference Newton method, simple Newton method, damped Newton method, trust region Newton method, etc [9].

Optimization method is to transform the nonlinear equations into an optimization problem, and then apply optimization theory to solve it [10-14]. The basic idea can be expressed in the following figure:

According to the different objective function, the conversion mode is also different, this paper summarizes and introduces the more common conversion mode in the following.

The optimization problem after transformation is mainly solved by some intelligent optimization algorithms. In this paper, three common algorithm technologies are summarized, and some new intelligent algorithms are briefly introduced, such as genetic algorithm [21, 22], simulated annealing algorithm [23, 24], particle swarm optimization algorithm [25], etc.

In order to investigate the variation of real solutions, interval iteration method [15, 16] has been proposed, which can find the variation interval of a real solution. However, due to the complexity of the calculation for solving the transcendental equations, the application of this method is limited. In addition, in order to reduce the requirements on the initial value of the iteration, the homotopy continuation algorithm and simplex method have also been proposed [17-19]. The homotopy continuation algorithm requires the Jacobi matrix norm of the equations to be bounded, which may be difficult to satisfy and verify, so there are still many problems to be solved when this method is applied to practical calculation. Simplex method, also known as fixed point algorithm or cyclic compensation algorithm, has the advantage of low sensitivity of initial value, avoiding the singularity of iterative method, and the disadvantage is that the calculation is complex and the calculation is large.

In this paper, the current research status of solving nonlinear equations with traditional iterative method and optimization method will be reviewed respectively, the research achievements of domestic and foreign scholars will be systematically reviewed, and the future research direction will be proposed.

2. THE ITERATIVE METHOD OF SOLVING NONLINEAR EQUATIONS

As a traditional optimization algorithm, the basic principle of the iterative method is to obtain the optimal solution step by step based on gradient information [27]. At present, the common algorithms are: Newton method, conjugate gradient method, fastest descent method, etc. Among them, Newton method and its derived quasi-Newton method and imprecise Newton method are the most classic, and also the most important methods to get the numerical solution of nonlinear equations. This paper mainly discusses the research status of Newton method and other improved algorithms derived from it.

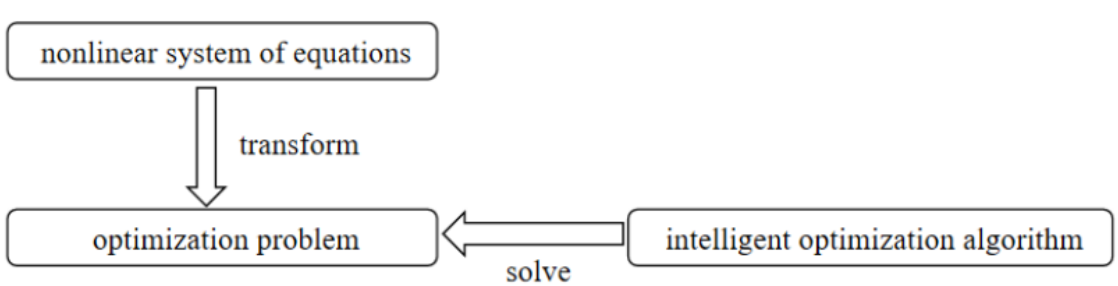

Figure 1: The idea of Optimization method to solve nonlinear equations.

2.1. The Basic Ideas of Newton Iteration Method

The basic idea of Newton iteration method is to carry out Taylor expansion of the nonlinear system of equations $F(x)=0$, and make linear approximation at the iteration point, so as to obtain:

$$F(x_k+s)\approx F(x_k)+F'(x_k)s=0 \tag{2.1}$$

x_k is the current iteration point, and $s=x-x_k$, if $F'(x_k)$ is non-singular, then:

$$x=x_k-F'(x_k)^{-1}F(x_k) \tag{2.2}$$

Given an initial value, an iterative format can be constructed:

$$x_{k+1}=x_k-F'(x_k)^{-1}F(x_k),k=1,2,\cdots \tag{2.3}$$

This is the classic Newton iteration. In the k step of the iteration, we need to solve the following Newton equation to obtain the step size of the iteration s_k:

$$F'(x_k)s_k=-F(x_k) \tag{2.4}$$

Newton iteration method has better local convergence, but it depends on the selection of initial points.If the initial point is selected not properly, it may not find the solution. At the same time, due to the calculation of gradient information, each step needs to calculate the Jacobian matrix and solve the linear equations. When the scale of the equations is large, the calculation amount is cumbersome, and is only applicable to the differential function, and it is easy to fall into the local optimal solution. Therefore, the improvement of Newton's method is mainly focused on solving or optimizing the above problems.

2.2. Improvement of Newton Iteration Method

2.2.1. Improvement in Algorithm Computation

Dembo *et al*. proposed an inexact Newton method for solving nonlinear equations. In each iteration of Newton's method, only the inexact solution of Newton's equation was carried out, and its local convergence under appropriate conditions was given [28]. On this basis, Eisenstat and Walker proposed a class of globally convergent inexact Newton method by combining line search method and trust region technique [29]. In fact, the inexact Newton method is a kind of internal and external iterative algorithm, the outer iteration is the traditional Newton method, and the inner iteration is usually some classical algorithms to solve linear equations, common SOR iteration, Gauss-Seidel iteration and Jacobian iteration. In recent years, with the rapid development of Krylov subspace method, Newton-Krylov subspace method [30, 31] has been widely used as a special method. This method uses Krylov subspace method as the inner iteration solver and Newton method as the outer iteration. This inner and outer iteration technique can make full use of Jacobian matrix, the structure and sparsity of Newtonian method greatly reduce the amount of computation. In 2021, Longquan Yong reviewed the second-order, third-order, fourth-order, fifth-order and ninth-order Newton methods for solving nonlinear equations [70], and gave the convergence efficiency index, finding that the convergence of various Newton methods is related to the selection of initial points, and the higher the convergence order, the greater the computational cost (memory occupied and function call times).

2.2.2. Improvement in Algorithm Convergence

Bai,Golub, and Ng first proposed an HSS iteration method to solve a positive definite linear system of equations with no Hermitian [32]. Subsequently, Bai and Guo took the HSS iteration method as the internal iteration solver of the inexact Newton method, established a Newton-HSS method for solving nonlinear equations with non-Hermitian positive definite Jacobian matrix, proved local and semi-local convergence, the numerical results showed that the Newton-HSS method was more efficient than the Newton Krylov subspace method [33]. In recent years, with the rapid development of HSS iterative method [34, 35], Aili Yang *et al.* proposed Newton-PSS method for solving nonlinear equations and gave a proof of convergence [36]. Xu Li used the GHSS and GPSS methods as the internal iterative solvers of the inexact Newton method respectively, proposed the Newton-GHSS method and the Newton-GPSS method for solving nonlinear equations, and gave their acceleration methods [37]. Recently, Wu and Chen proposed the modified Newton-HSS method instead of the classical Newton method as the external iterative solver of the inexact Newton method, while the HSS method was used as the internal iterative solver, and proved the local and semi-local convergence of the method under appropriate conditions. The numerical results show the modification The Newton-HSS method is superior to the Newton-HSS method in terms of iteration steps and CPU time [38, 39]. Dai *et al*. have successively proposed the revised Newton-NSS method and the revised Newton-PSS method [40, 41].

Not long ago,Li and Guo used the multi-step modified Newton-HSS method as the external iterative solver of the inexact Newton method to establish the multi-step modified Newton-HSS method for solving nonlinear equations, and proved the local and semi-local convergence of the method. Numerical results show that the multi-step modified Newton-HSS method is more effective than the modified Newton-HSS method [42].

2.2.3. Improvement in Other Aspects

In addition, in order to avoid the waste of point information of the first two iterations and the serious problem of nonlinear approximation function caused by excessive use of information, Yongkang Sui *et al.* proposed a rational approximation with linear numerator and denominator based on the information of the last two iterations, and proposed a RALND function [43, 44]. Subsequently, scholars did further research on this basis and produced a large number of optimization methods and solving methods for nonlinear equations [45, 46]. Saheya *et al*. constructed an improved RALND function and proposed a class of quasi-Newton method based on RALND function [47]. Wang applied RALND function to nonlinear least squares problem and proposed a class of improved Gauss-Newton method based on RALND function [46].

Although Newton's method and inexact Newton's method are very effective algorithms to solve nonlinear problems, but they both need to calculate and store matrices, which make it more difficult to solve large problems. In reference [71] Cruz and Raydan proposed a non-monotonic spectral gradient method for solving general nonlinear equations and proved its global convergence. Both of these two spectral gradient methods are suitable for solving large-scale interval problems.

3. THE ITERATIVE METHOD OF SOLVING NONLINEAR EQUATIONS

The solution of using optimization method to solve nonlinear equations can be roughly divided into two steps [47]:

(1) Transform it into an optimization problem (*e.g.*, single-objective optimization, constrained optimization, multi-objective optimization, etc.);

(2) Intelligent optimization algorithm is used to solve the optimization problem after transformation. It is worth noting that the optimization problem after transformed has the same optimal solution as the original nonlinear equations.

3.1. Transformation Method from Nonlinear Problem to Optimization Problem

According to the number of objective functions and the absence of constraints, researchers put forward three types of methods to transform nonlinear problems into optimization problems, namely, single-objective transformation method, constrained transformation method and multi-objective transformation method [47].

(1) The single objective transformation method is to transform the nonlinear problem into a model that minimizes the sum of squares or absolute value. It is widely used because it is simple and easy to execute.

(2) The intelligent optimization algorithm is essentially an unconstrained search algorithm actually, and it needs to be combined with some constraint processing techniques to solve some constraint optimization problems, so there are relatively few applications and corresponding researches.

(3) The multi-objective transformation method is to transform the nonlinear problem into a multi-objective optimization problem, and the relationship between multiple objective functions is mutually exclusive. In 2008, Grosan proposed a CA transformation method, which regarded each nonlinear equation as an objective function. This method is simple and intuitive, but with the increase of the number of equations, the performance of the algorithm will be significantly reduced [48]. To solve this problem, Song proposed a MONES(multiobjective optimization for nonlinear equation systems) transformation method which transforms the nonlinear problem into two objective functions by optimizing the two optimal functions simultaneously, the calculation difficulty is greatly reduced [49]. However, because only one dimension of information is considered, some optimal solutions will be lost in the optimization process. To this end, Gong proposed the A-Web algorithm in 2017 to assign weights to each decision vector and improve the global search performance [50].

To sum up, the single objective transformation method and the constrained transformation method may not find multiple optimal solutions in the optimization process due to the single objective, and attention should be paid to increasing diversity when designing optimization algorithms. The multi-objective transformation method can find multiple optimal solutions in one run.

3.2. Intelligent Algorithm for Solving Optimization Problems

The reason why intelligent optimization algorithm is used to solve nonlinear problems is that it is a crowd-based optimization algorithm, which searches from multiple points at the same time, rather than a single point search, and has invisible parallelism. Moreover, the initial point requirements are not high, and it is still applicable to nonlinear equations without differentiability, with a wide range of solutions, high efficiency, robustness and other characteristics, which greatly facilitates the solution of nonlinear equations and solves the limitations of traditional optimization algorithms. Therefore, the use of intelligent optimization algorithms to solve nonlinear equations has attracted more and more attention, and has become a research hotspot in recent years.

According to recent research trends, they can be divided into four categories: algorithms based on exclusion technology, algorithms based on clustering technology, algorithms based on multi-objective technology, and other intelligent algorithms [47].

3.2.1. Algorithm Based on Exclusion Technology

Exclusion technique is a common method for solving multiple optimal solutions, its main principle is as follows: design a function based on the objective function, set an exclusion region near the discovered optimal solution, and force the algorithm to continue searching for the optimal solution in other regions, so as to explore new search regions and discover new optimal solutions [51].

At present, common repulsion techniques include multiplicative repulsion technique and additive repulsion technique.

(1) multiplicative repulsion technique

Pourjafari *et al*. [52] proposed the multiplication exclusion technique in the following form:

$$\min R(x) = (f(x)+\varepsilon)\sum_{j=1}^{K}\left|\coth(\alpha\left\|x-x_j^*\right\|)\right| \tag{3.1}$$

where K is the number of optimal solutions that have been found, ε is a normal number close to zero, α is the exclusion radius, which is used to adjust the size of the exclusion region, x_j^* is the j and optimal solution that has been found.

Ramadas *et al*. [53] proposed another multiplicative exclusion technique based on error function "erf", in the following form:

$$\min R(x) = (f(x)+\varepsilon)\sum_{j=1}^{K}\varsigma_{\rho'}(\gamma,\left\|x-x_j^*\right\|) \tag{3.2}$$

where γ is the measure of the degree of repulsion, ρ' is the radius of repulsion.

(2) additive repulsion technique

Hirsch *et al*. [54] put forward the addition exclusion technique to solve nonlinear problems in the following form:

$$\min R(x) = f(x)+\beta\sum_{j=1}^{K}e^{-\left\|x-x_j^*\right\|}\chi_\rho(\left\|x-x_j^*\right\|) \tag{3.3}$$

where χ_ρ represents the characteristic function, ρ is a small constant that adjusts the radius of repulsion, β is a measure of rejection.

In literature [51], a new optimization algorithm RADE(repulsion-based adaptive differential evolution) is proposed by combining the addition exclusion technique with the difference algorithm (differential evolution, DE) [55] to solve nonlinear problems. After finding an optimal solution, the algorithm will use the addition exclusion technique to adjust the objective function, increase the value of the function near the optimal solution, make it become the exclusion region, and force the algorithm to continue searching for a new region and find other optimal solutions.

At the same time, in order to increase the diversity of selection, neighborhood variation and crowded selection are added on the basis of the original difference method, and the adaptive parameter adjustment strategy can improve the search performance of RADE algorithm.

When using repulsion technique to solve nonlinear problems, the repulsion radius plays an important role. If the repulsion radius is set too large, the optimal solution that is closer to the found optimal solution may be lost. If the setting is too small, the algorithm's search efficiency will be greatly reduced. Liao put forward an evolutionary algorithm based on dynamic exclusion in reference [56]. In order to improve the search efficiency of the algorithm at the beginning of the algorithm, a large exclusion radius is set for global search. As the number of iterations increases, the exclusion radius is gradually reduced so that a new optimal solution near the optimal solution can be found. Secondly, a dynamic repulsion based evolutionary algorithm (DREA) framework is proposed, which can be combined with different evolutionary algorithms and repulsion techniques to greatly improve the probability of finding the optimal solution.

3.2.2. Algorithm Based on Clustering Technology

The so-called clustering technique is to classify the solutions of nonlinear equations into different categories. The solutions after classification have high similarity, and then search the optimal root from each category.

Tsoulos *et al*. [57] proposed a global search algorithm based on clustering to solve nonlinear equations. Firstly, the nonlinear equation is transformed into a sum of squares single-objective problem, and then the clustering algorithm Multistart and Minfinder are used to solve the transformed problem. Sacco *et al*. [58] combined fuzzy clustering means (FCM) to classify the candidate solutions, and finally used N-M local search algorithm to search each class to accelerate the convergence of the population. He *et al*. [59] proposed the FNODE (fuzzy neighborhood based di*ff*erential evolution with orientation) algorithm, by improving fuzzy neighborhood technology, suitable individuals are selected according to fuzzy rules and individual distribution to form neighborhood, and the exploration ability of population is improved. Liao *et al*. [60] have proposed DDE/R(decomposition-based di*ff*erential evolution with reinitialization) algorithm and MENI-EA(memetic niching-based evolutionary algorithms) algorithm, which main idea is to classify the population, increase the diversity of the population, and search for new roots more better.

Although the clustering technology is easy to implement, but it is still depends on the selection of the number of clusters. If the number is selected too much, the search is still carried out even if the optimal solution is not used in some classes, resulting in a larger computation quantity. If the number selection is too small, only one root can be found in each class, some roots may be lost.

3.2.3. Algorithm Based on Multi-Objective Technology

Multi-objective technique is one of the commonly used algorithms to deal with multi-objective optimization problems [61], and its purpose is to find a set of Pareto optimal solutions. In recent years, a lot of research results of NESs problem solving using multi-objective technique have appeared. In general, they can be divided into two categories:

(1) Use Pareto to sort and select individuals;

(2) Divide the population into several subpopulations, each of which corresponds to an objective function, this mechanism is called the population-based approach.

Song *et al*. [49] proposed the MONES algorithm, in which the nonlinear problem was treated as a two-objective optimization problem, and the transformation problem was solved through NSGA-II. After the population is randomly initialized, the progeny population is generated through selection, crossover and variation. Thene the parents and children are combined, and individuals entering the next generation are selected using non-dominant sorting and crowding distance selection. Gong *et al*. [50] further improved MONES and proposed A-WEB(A weighted biobjective) optimization algorithm. In this algorithm, the weights in the objective function are randomly generated from 0 to 1. In the optimization process, two search algorithms, SHADE and NSGA-II, are combined to generate offspring through mutation and cross. In this process, the parameters are adjusted adaptively, thus improving the accuracy of the search. Then the choice between individuals is determined by the non-dominant order.

Gao *et al*. [62] designed a two-phase evolutionary algorithm (TPEA) to solve nonlinear problems. In this algorithm, the nonlinear problem is transformed into a single objective optimization problem. In the first stage, the niches technology NCDE was used to construct the diversity index based on Gaussian kernel function to maintain the diversity of the population and achieve the balance between convergence and diversity. In specific iterations, NCDE alternates with NSGA-II to produce high-quality candidate solutions. In the second stage, a probe method is designed to locate the promising region (*i.e*. the region where the best solution may exist), using DE as a local search algorithm to finally find the root of the nonlinear problem.

3.2.4. Other Intelligent Algorithms

(1) genetic algorithms (GA)

Literature [63] combines the Legendre numerical integration and GA algorithm to solve nonlinear problems. In literature [64], immune genetic algorithm (IGA) was used to find the roots, and individual distance comparison method based on fitness value was adopted, which not only retained superior individuals, but also reduced the selection of similar individuals, thus ensuring the diversity of the whole population.

Literature [65] proposes a hybrid niche genetic algorithm and quasi-Newton algorithm to solve nonlinear problems. Firstly, a population is generated by the microbiotic genetic algorithm, and the individual with the best fitness value is found. Then, an individual in the remaining population is selected as the initial point with a certain probability, and local search is carried out by quasi-Newton method. If the locally optimal individual fitness value is better than before, it is replaced, otherwise, no replacement operation is performed.

(2) particle swarm optimization (PSO)

Compared with GA algorithm, PSO algorithm is an algorithm with memory, good individual information will be saved and utilized, and in most cases, the convergence speed is faster than GA algorithm. The application of PSO algorithm to nonlinear problems has been widely studied. Mo *et al*. [66] proposed a conjugate direction particle swarm optimization (CDPSO) algorithm and applied the conjugate direction method to PSO to improve the local optimal problem that PSO is prone to fall into in high dimensions, thus facilitating the solution of nonlinear problems. Brits *et al*. [67] further improved PSO, proposed the operation of nbest (neighborhood best), increased the diversity of the algorithm, and made the algorithm easy to locate multiple roots of NESs.

(3) simulated annealing algorithm (SA)

In literature [68], in order to avoid premature convergence of SA algorithm and local optimal solution, fuzzy adaptive SA algorithm is used to overcome this defect when searching for multiple optimal solutions. Literature [69] proposes an polarization technique, similar to the repulsion technique, which continuously modifies the objective function after determining a new optimal solution to generate a repulsion region in the neighborhood of the previously found root, which solves the problem of SA algorithm falling into local optimality.

In addition, there are some new intelligent algorithms, such as neural network method [20] and ant colony algorithm [24], which will not be introduced in this paper.

4. CONCLUSIONS

At present, both iterative method and intelligent algorithm for solving nonlinear equations have reached a certain stage of maturity. Scholars at home and abroad have made a lot of optimization and improvement in the convergence and convergence speed of algorithms. But there are still some problems to be solved, and the future research must be focused on this:

(1) In practical application, such as the gear-meshing problem, the influence of relevant parameters on the solution of nonlinear equations and the practical significance of the specific solution need to be discussed in depth.

(2) For nonlinear equations with multiple solutions, how to determine appropriate iterative initial values in order to obtain all solutions still needs to be studied.

(3) At present, the theoretical research results on the numerical solution of nonlinear equations have been quite rich, but how to apply the research results to practical engineering projects needs to be further expanded and verified.

(4) For large-scale nonlinear equations without constraints, how to accelerate the convergence speed and ensure that all optimal solutions are obtained needs to be further optimized.

(5) For the study of the existence and initial value sensitivity of the real solutions, most of them are carried out by using the principles of compression mapping and fixed point in nonlinear functional analysis and the theory of convex sets and convex functions, but the conclusions and discriminant conditions obtained are difficult to apply in concrete calculations.

CONFLICT OF INTEREST

No conflict of interest

REFERENCES

[1] J.M. Ortega, W.C. Rheinboldt, Jujina (Chinese translation). Iterative Solution of multivariate nonlinear Equations, Beijing: Science Press, 1983.

[2] D. Wang. Solution and Optimization of Nonlinear Equations, Beijing: People's Education Press, 1983.

[3] R. Xie, P. Jiang. Nonlinear Numerical Analysis, Shanghai: Shanghai Jiao Tong University Press, 1984.

[4] D.M. Young, J.F. Traub. Iterative methods for the solution of equations [M]. American Mathematical Monthly, 1967. https://doi.org/10.2307/2316079

[5] M.S. Petkovi', B. Neta, Multipoint methods for solving nonlinear equations [J]. Applied Mathematics and Computation, 2014, 226(1): 635-660. https://doi.org/10.1016/j.amc.2013.10.072

[6] Q. Ni. Optimization Method and Program Design [M]. Wuhan: Science Press, 2009.

[7] G. Feng. Iterative Solution of Nonlinear Equations [M]. Shanghai: Science and Technology Press, 1989.

[8] W.C. Rheinboldt. Methods for solving systems of nonlinear equations [M]. SIAM, 1998. https://doi.org/10.1137/1.9781611970012

[9] J. Moser. Stable and random motions in dynamical systems: With special emphasis on celestial mechanics [M]. Princeton University Press, 1973.

[10] N. Deng *et al*. Computational methods for Unconstrained Optimization, Beijing: Science Press, 1982.

[11] J. Xu. Mechanical Optimization Design, Beijing: China Machine Press, 1996.

[12] W. Liu. Mechanical Optimization Design (2nd Edition), Beijing: Tsinghua University Press, 1997.

[13] Jorge Nocedal, Stephen J. Wright. Numerical Optimization, Springer Science + Business Media, Inc., 1999. https://doi.org/10.1007/b98874

[14] B. Chen. Optimization Theory and Algorithm, Beijing: Tsinghua University Press, 2000.

[15] Ramon E. Moore, Methods and Application of Interval Analysis, SIAM, Philadelphia, Pennsylvania, 1979.

[16] D. Wang, L. Zhang, Deng Nai-yang, Interval Algorithms for Nonlinear Equations (Computational Mathematics Series), Shanghai: Shanghai Science and Technology Press, 1987.

[17] Q. Li, Z. Mo, L.Qi. Numerical Solution of Nonlinear Equations, Beijing: Science Press, 1999

[18] Editorial Board, Handbook of Modern Applied Mathematics. Handbook of Modern Applied Mathematics: Calculation and Numerical Analysis Volume, Beijing: Tsinghua University Press, 2005.

[19] Z. Wang. A discussion on Homotopy methods, Dalian: Dalian University of Technology Press, 2011.

[20] Y. He, X. Li. Neural Network Control Technology and its Application, Beijing: Science Press, 2000.

[21] X. Wang, L. Cao. Genetic Algorithm Theory, Application and Software Implementation, Xi 'an: Xi 'an Jiaotong University Press, 2004.

[22] M. Gen, R. Cheng, Yu X *et al*. Genetic Algorithm and Engineering Optimization, Beijing: Tsinghua University Press, 2005.

[23] L. Kang, Y. Xie, Y. You, *et al*. Nonlinear Numerical Parallel Algorithm Simulated Annealing Algorithm (Volume 1), Beijing: Science Press, 2003.

[24] W. Xing, J. Xie. Modern Optimization Calculation Methods, Beijing: Tsinghua University Press, 2005.

[25] Y. Zhang, D. Gong. Advanced Multi-objective Particle Swarm Optimization Theory and its Application, Beijing: Science Press, 2016.

[26] R. Horst, P. M. Pardalos, N. V. Thoai, Yellow and Red Selection (Chinese translation). Introduction to Global Optimization, Beijing: Tsinghua University Press, 2005.

[27] Broyden C G. A class of methods for solving nonlinear simultaneous equations [J]. Mathematics of Computation, 1965, 12 (92): 577-593. https://doi.org/10.1090/S0025-5718-1965-0198670-6

[28] R.S. Dembo, S.C. Eisenstat, T. Steihaug. The Inexact Newton methods [J]. Journal of SIAM Journal on Numerical Analysis, 1982, 12 (2): 400-408. https://doi.org/10.1137/0719025

[29] S.C. Eisenstat, H.F. Watker. Globally convergent inexact Newton methods [J]. SIAM Journal on Optimization, 1994, 4 (2): 393-422. https://doi.org/10.1137/0804022

[30] P.N. Brawn, Y. Saad. Hybrid Krylov methods for nonlinear systems of equations [J]. SIAM Journal on Scientific and Statistical Computing, 1990,11(3): 450-481. https://doi.org/10.1137/0911026

[31] P.N. Brawn, Y.Saad. Convergence theory of nonlinear Newton-Krylov algorithms [J]. SIAM Journal on Optimization, 1994, 4 (2): 297-330. https://doi.org/10.1137/0804017

[32] Z.Z. Bai, G.H. Golub, M.K. Ng. Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear [J]. SIAM Journal on Matrix Analysis and Applications, 2003,24: 603-626. https://doi.org/10.1137/S0895479801395458

[33] Z.Z. Bai, X.P. Guo. On Newton-HSS methods for systems of nonlinear equations with positive definite Jacobian matrices [J]. Journal of computational mathematics, 2010, 28(2): 235-260. https://doi.org/10.4208/jcm.2009.10-m2836

[34] Y. Cao, W.W. Tan, M.Q. Jiang. A generalization of the positive-definite and skew-Hermitian splitting Numerical Algebra Control and Optimization, 2012, 2(4): 811-821. https://doi.org/10.3934/naco.2012.2.811

[35] M. Benzi, D. Bertaccini. A generalization of Hermitian and skew-Hermitian splitting iteration [J]. SIAM Journal on Matrix Analysis and Applications, 2009,31(2): 360-374. https://doi.org/10.1137/080723181

[36] A. Yang, Y. Wu, X. Li, L. Meng. Newton-PSS iterative method for a Class of nonlinear equations [J]. Computational Mathematics, 2014, 4: 3-14.

[37] LI X. Research on HSS based iterative method and acceleration technique for several linear and nonlinear equations and a class of continuous Sylverster equations [D]. PhD Thesis, Lanzhou University, 2013.

[38] Q.B. Wu, M.H. Chen. Convergence analysis of modified Newton-HSS methods for systems of nonlinear equations [J]. Numerical Algorithms, 2013: 659-683. https://doi.org/10.1007/s11075-012-9684-5

[39] M.T. Darvishi, A. Barati. A thrid-order Newton-type method to solve systems of nonlinear equations [J].Applied Mathematics and Computation, 2007, 187: 630-635. https://doi.org/10.1016/j.amc.2006.08.080

[40] P.F. Dai, Q.B. Wu, M.H.Chen. Modified Newton-NSS method for solving systems of nonlinear equations [J]. Numerical Algorithms, 2018, 77: 1-21. https://doi.org/10.1007/s11075-017-0301-5

[41] P.F. Dai, Q.B. Wu, Y.X. Wu, W.L. Liu. Modified Newton-PSS method to solve nonlinear equations [J]. Applied Mathematics Letters, 2018, 86: 305-312. https://doi.org/10.1016/j.aml.2018.07.004

[42] Y. Li, Guo. Y. Multi-step modified Newton-HSS methods for systems of nonlinear equations with positive definite Jacobian matrices [J]. Numerical Algorithms, 2017, 75: 55-80. https://doi.org/10.1007/s11075-016-0196-6

[43] Sui Yunkang, Ye B. A convenient and practical rational approximation and its improvement on a large number of optimization methods [J]. Journal of Operations Research, 1993, 12(1): 52-66.

[44] Sui Y, Sa Heya, Chen Guoqing. Improvement of Rational approximation RALND for cumulative two-point information [J]. Computational Mathematics, 2014, 36(1): 51-64.

[45] Sui Y, Yang Z. Improved Newton Method and Dual Method using two-point rational approximation [J]. Journal of Dalian University of Technology, 1994, 12(1): 2-9.

[46] H.Y. Sa, G.Q. Chen, Y.k. Sui, C.Y. Wu. A new Newton-like method for solving nonlinear Equations [J]. SpringerPlus, 2016, 5 (1): 1269. https://doi.org/10.1186/s40064-016-2909-7

[47] Gao W, Luo Y. Review of intelligent optimization algorithms for solving nonlinear equations [J]. Control and Decision, 2019, 36(4): 769-778.

[48] Grosan C, Abraham A. A new approach for solving nonlinear equations systems [J]. IEEE Transactions on Systems, Man and Cybernetics-Part A, 2008, 38(3): 698-724. https://doi.org/10.1109/TSMCA.2008.918599

[49] Song W, Wang Y, Li H X, *et al*. Locating multiple optimal solutions of nonlinear equation systems based on multiobjective IEEE Transactions on Evolutionary Computation, 2015, 19(3): 414-431. https://doi.org/10.1109/TEVC.2014.2336865

[50] Gong W Y, Wang Y, Cai Z H. A weighted biobjective transformation technique for locating multiple optimal solutions of nonlinear equation systems[J]. IEEE Transactions on Evolutionary Computation, 2017, 21(5): 697-713. https://doi.org/10.1109/TEVC.2017.2670779

[51] Gong W Y, Wang Y, Cai Z H. Finding multiple roots of nonlinear equation systems via a repulsion-based adaptive differential evolution [J]. IEEE Transactions on Systems, Man, and Cybernetics: Systems, 2020, 50(4): 1499-1513 https://doi.org/10.1109/TSMC.2018.2828018

[52] Pourjafari E, Mojallali H. Solving nonlinear equations systems with a new approach based on invasive weed optimization algorithm and clustering [J]. Swarm and Evolutionary Computation, 2012, 4: 33-43. https://doi.org/10.1016/j.swevo.2011.12.001

[53] Ramadas G C, Fernandes E M, Rocha A A. Multiple roots of systems of equations by repulsion merit functions [J]. Computational Science and Its Applications, 2014(2): 126-139. https://doi.org/10.1007/978-3-319-09129-7_10

[54] Hirsch M J, Pardalos P M, Resende M G C. Solving systems of nonlinear equations with continuous GRASP [J]. Nonlinear Analysis: Real World Applications, 2009, 10(4): 2000-2006. https://doi.org/10.1016/j.nonrwa.2008.03.006

[55] Storn R, Price K. Differential evolution-A simple and efficient heuristic for global optimization over continuous spaces. Journal of Global Optimization, 1997, 11(4): 341-359. (in Chinese) https://doi.org/10.1023/A:1008202821328

[56] Liao Z W, Gong W Y, Yan X S. Solving nonlinear equations system with dynamic repulsion-based evolutionary Algorithms [J]. IEEE the Transactions on Systems, Man, and Cybernetics, Systems, 2020, 50 (4): 1590-1601. https://doi.org/10.1109/TSMC.2018.2852798

[57] Tsoulos I G, Stavrakoudis A. On locating all roots of systems of nonlinear equations inside bounded domain using global optimization methods [J]. Nonlinear Analysis: Real World Application, 2010,11(4): 2465-2471. https://doi.org/10.1016/j.nonrwa.2009.08.003

[58] Sacco W F, Henderson N. Finding all solutions of nonlinear systems using a hybrid metaheuristic with fuzzy clustering means [J]. Applied Soft Computing, 2011, 11(8): 5424-5432. https://doi.org/10.1016/j.asoc.2011.05.016

[59] He W, Gong W Y, Wang L, *et al*. Fuzzy neighborhood based differential evolution with orientation for nonlinear equation system [J]. Knowledge-Based Systems, 2019, 182: 104796. https://doi.org/10.1016/j.knosys.2019.06.004

[60] Liao Z, Gong W, A decomposition based differential evolution with reinitialization for nonlinear equations systems [J]. Knowledge-Based Systems, 2019, 191: 105312. https://doi.org/10.1016/j.knosys.2019.105312

[61] Zhang Q F, Li H. A multi-objective evolutionary algorithm based on decomposition [J]. IEEE Transactions on Evolutionary Computation, 2007, 11(6): 712-731. https://doi.org/10.1109/TEVC.2007.892759

[62] Gao W F, Li G H, Zhang Q F, Solving nonlinear equation systems by a two-phase evolutionary algorithm [J]. IEEE Transactions on Systems, Man, and Cybernetics: Systems, 2019: 1-12.

[63] Mhetre P S. Genetic algorithm for linear and nonlinear equation [J]. International Journal of Advanced Engineering Technology, 2012(2): 114-118.

[64] Zhou L, Jiang C S. New method for solving nonlinear equation systems [J]. Journal of Chinese Conputer Systems, 2008, 29(9): 1709-1713.

[65] Kennedy J, Eberhart R C. Particle swarm optimization [C]. Proceedings of the 1995 IEEE International Conference on Neural Networks. Perth.

[66] Wang J. Immune genetic algorithm for solving nonlinear equation[C]. Proceedings of the 2011 International Conference on Mechatronic Science, Electric Engineering and Computer. Jilin, 2011: 2094-2097.

[67] Brits R, Engelbrecht A P. Solving systems of unconstrained equations using particle swarm optimization[C]. Proceedings of the 2002 IEEE International Conference Systems, Man and Cybernetics. Yasmine Hammamet, 2002: 6-9.

[68] Oliveira H, Petraglia A. Solving nonlinear systems of functional equations with fuzzy adaptive simulated annealing [J]. Applied Soft Computing, 2013, 13(11): 4349-4357. https://doi.org/10.1016/j.asoc.2013.06.018

[69] Henderson N, Sacco W F, Platt G M. Finding more than one root of nonlinear equations via a polarization technique: An application to double retrograde vaporization [J]. Chemical Engineering Research& Design, 2010, 88(5): 551-561. https://doi.org/10.1016/j.cherd.2009.11.001

[70] L. Yong. Research progress of Newton iteration method for nonlinear equations [J]. Mathematics in Practice and Understanding, 2021, (15).

[71] Cruz W, Raydan M, Nonmonotone speetral methods for large-seale nonlinear systems. Optimization Methodsand Software 18 2003 (5): 583-559. https://doi.org/10.1080/10556780310001610493

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